1. Stochastic Processes and Stochastic Calculus

%%% Kim’s comments

In the microscopic motion of molecular its position is measureable so that

Now the velocity of the molecular is

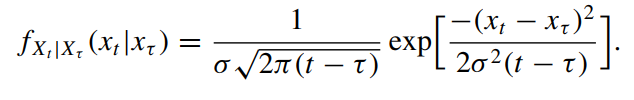
* If the noise is a Brownian motion, is not defined at any time.

In fact if you measure something using a continuous sensor, the noise is look like a Brownian motion. Hence the derivative of the output of the sensor is not defined.  
Engineers are approximated this as a white noise even if the derivative is not exist.

So that

%%%%

* 1. Random Walk and Brownian Motion
* Def. 5.1 A scalar Brownian motion process is defined as a process such that

1. is a Gaussian random variable
2. has independent increments
3. 

* Transition probability
  1. Mean Square Calculus

%% Kim: Introduce differential equations in continuous stochastic process

1. Convergent, limit : In deterministic

If , there is a such that

1. If is a random variable, function, how to define the convergence?

%%

* Def. 5.3 A function is continuous at a point,, if and only if
* Def.5.4 A function is differentiable at a point,, if and only if

%%% What is the definition of Integral? There are several definition of integral as

1. Riemann Integral
2. Ito stochastic integral,
3. Stratonovich integral

So on. Sometimes their integral are equal to each other and sometimes are different. %%

* Def. 5.5 random variable sequence converges to

in probability

with probability 1

in the mean square sense :

* Correlation , covariance function in

-

-correlation :

-covariance:

* Mean square derivatives (5.16)

* Theorem 5.12

If is mean square continuous at if and only if is continuous at the diagonal point

* Theorem 5.14 : is mean square differentiable at iff exists at
* Example 5.17 : Brownian Motion is mean-square continuous

Proof: The correlation of Brownian is so that if ,

which is continuous , which implies the Brownian is continuous in mean square sense

* **Brownian is not differentiable : Important fact in stochastic differential, Brownian motion.**

Which implies for any Brownian is not differentiable !!

* **White noise : Engineer assumption:**

**White noise, is a derivative of Brownian by definition as**

1. The two are independent if
2. E[W(t)] = 0

%% Kim’s comment : independent of Random variables

Recall the definition of the independent:

1. If the events are independent if
2. If two pdfs is independent if the multivariate joint pdf
3. Two Gaussians is independent if i.e., the covariance matrix is a diagonal.
4. The Brownian motion is an independent increment process, i.e.,

and are independent if and are disjoint

1. The white noise is an independent process %%%

%% Kim’s comment – orthogonal

Two random variables are orthogonal if

%%%

* 1. Wiener Integrals - skip
  2. integral

%% Kim’s comment

Consider the following differential equation as

The solution is

Another notation as

Then the solution is

since

Hence the solution is equivalent to each other !!

Now the input is a Brownian motion, which is not differentiable, so mathematician use a different notation as (c.2). However engineers prefer (c.1) as

So that

Where

even if the derivative of Brownian does not exist but it is assumed as a white noise %%

Consider **a stochastic differential equation**

where

is defined in terms of its integral representation as

* stochastic integral:

%%% Kim’s comment: Riemann Integral

<https://en.wikipedia.org/wiki/Riemann_integral>

A partition of an interval as as

The sum is defined as

1. Riemann Integral

Define

If , , then Riemann value of

1. Ito integral

Let

1. The Stratonovich Integral

Let ,

Looks simpler than Riemann? Maybe. The problem is happen if x is Brownian, since

Due to the fact

So that if is Brownian, it is not Riemann integrable. %%%

* Example 5.20 – strange fact…(I put

I =

Proof:

By the definition of Ito integrtal

The second term is .

The first term is a little bit strange ….

Since

Then taking expectation on it gives

Hence in the limit, the first term **converges to**  in mean square sense, hence in the mean square sense

%% Kim’s comment:

If is not a Brownian motion, consider hen

In (5.40) the value of integral has another component as

* 1. Second order Ito integral 🡪 skip
  2. Stochastic Differential Equations and Exponentials 🡪 skip
  3. The Ito Stochastic Differential
* Theorem 5.23

Let be the unique solution to the vector Ito stochastic differential equation,

and .

Let be a scalar-valued real function of that is continuously differentiable in and that has continuous second derivatives with respect to . The stochastic differential of is then

* Example. 5.24 : Consider

Calculate

Sol: Let so that

Using (5.49)

Substitute (5.51) into (5.52)

Take the Expectation on both sides

Hence

%%% Kim’s comment

Consider Eq.5.52, in an ordinary differential equation,

**However, the RHS of Eq.(5.52) is different!!**

%%%%

* 1. **Continuous – Time Gauss-Markov Processes**

Consider continuous time Gauss-Markov process(linear system)

1. **The mean of ,**
2. The correlation of

Proof: (Using Theorem 5.23 in the vector case)

Define

Then by theorem 5.23

Since

And

Take the expectation, then

Now the increment of Brownian is independent any other random process,

So that

Denote , then

QED

1. **The error covariance of**

Proof :

Let . Then

Denote , as the same procedure for the correlation,

%%% Kim’s comments : engineer’s method

Consider

The solution of (e.1) is

Now the mean is

which is equivalent to the solution of (5.58)

Now the error covariance is

Differentiate at both sides

-QED -

%% Kim’s comment:

One more a strange formula as (5.24). Since

Hence

Which is different from the previous result. But this is not correct since if is the solution to the SDE with white noise, so that it is not allowed

Compared to the ordinary differential chain rule as

Check it with (5.24)

QED %%%

%% Kim’s comment:

In linear system, in order to check the stability, one method is Lyapunov.

Given a dynamic system

If there is a , the solution of the following Lyapnov equation

is a positive definite, , then the system is asymptotically stable.

Now the steady state of eq.(5.59) is

Compare a) and b), they are similar to each other but the order is different.

And the solutions are different even if they are positive definite.

%%%%%%%%